LP property for C*-algebras

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(joint work with Dinh Trung Hoa and Toan Minh Ho)

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Motivation

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If the Jones index of a II_1 factor N is finite with respect to a II_1-factor M, then M and N share many common properties such that as hyperfiniteness, property Γ, property T, as shown by [Pimsner and Popa:86].
In the case of inclusion of simple C*-algebras with a finite Watatani index we could not hope such an thing. In fact there are examples of inclusion $\mathrm{CAR} \subset \mathrm{CAR} \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ such that $\mathrm{CAR} \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ are not AF by [Blackadar:90] and [Elliott:89].

Related to the classification theorem Zacharias and Winter (2010) conjectured the following;

Conjecture

For a separable, finite, nonelementary, simple, unital and nuclear C*-algebra A, the followig properties are equivalent:

1. A has finite nuclear dimension.
2. A is $\mathcal{Z}$-stable.
3. A has strict comparison of positive elements.
4. A has almost unperforated Cuntz semigroup.

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It is known that (2) $\rightarrow$ (3) $\leftrightarrow$ (4), cf. [Rordam:1992] and [Rordam:2004], and (1) $\rightarrow$ (2), cf. [Winter:2011].

AH-algebras

AH-algebras

Definition

A C*-algebra A is called an *AH algebra* if A is an inductive limits of C*a-lgebras A_n, where A_n have the form

$$A_n = \oplus_{j=1}^{r_n} M_j(C(X_{n,j}))$$

with each $X_{n,j}$ a compact metrizable space, r_n finite and M_j the C*-algebra of $j \times j$ complex matrices.

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Theorem (Winter 2011)

Let A be a separable, simple, nonelementary, unital, AH algebra. T.F.A.E

1. A has slow dimension growth.
2. A is $\mathcal{Z}$-stable.
3. A has finite decomposition rank.

Small eigenvalue variation

Small eigenvalue variation

Definition (Bratteli-Elliott '96)

Let B be a C*-algebra such that

$$B = \oplus_{i=1}^{k} M_{n_i}(C(X_i)),$$

where X_i is a connected compact Hausdorff space for ecvery i. Let a be any self-adjoint element in B. For any $x \in X_i$, any positive inetger m $(1 \leq m \leq n_i)$, let λ_m denote the mth lowest eigenvalue of $a(x)$ counted with multiplicity.

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Then the variation of eigenvalues of a, denoted $EV(a)$, is defined as the maximum of the nonnegative real numbers

$$\sup\{|\lambda_m(x) - \lambda_m(y)|; x, y \in X_i\},$$

over all i and all possible values of m.

Definition (Bratteli-Elliott '96)

An AH algebra B of homogeneous C*-algebra B_i with morphisms ϕ_{ij} from B_i to B_j. Suppose that

$$B_i = \oplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it})),$$

where k_i and n_{it} are positive integers and X_{it} is a connected compact Hausdorff space for every positive integer i and $1 \leq t \leq k_i$.

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Then B is said to have *small eigenvalue variation* if for any self-adjoint element a in B_i, and any positive number $\varepsilon > 0$, there is a $j \geq i$ such that

$$EV(\phi_{ij}(a)) < \varepsilon.$$

Theorem (Blackadar-Dâdârlat-Rørdam '91, Blackadar-Bratteli-Elliott-Kumujian '92,Bratteli-Elliott '96)

Let B be an AH algebra in the definition. Consider the following conditions.

(1) The projections of B separates the traces on B.

(2) B has small eiganvalue variation.

(3) B has real rank zero.

(i) The following implications hold in general:

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$$(3) \to (2) \to (1).$$

(ii) If B is simple, then $(1) \Leftrightarrow (2)$.

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(ii) If B is simple, then (1) $\Leftrightarrow$ (2).

(iii) If B is simple and has slow dimension growth, then the conditions (1), (2) and (3) are equivalent.

Diagonal AH algebras

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Definition

(1) Let X, Y be compact Hausdorff spaces. A *-homomorphism ϕ from $M_m(C(X))$ to $M_{nm}(C(Y))$ is said to be diagonal if there exist continuous maps $\{\lambda_i\}_{i=1}^m$ from Y to X such that

$$\phi(f) = \mathrm{diag}(f \circ \lambda_1, f \circ \lambda_2, \ldots, f \circ \lambda_n) \ (f \in C(X, M_m)).$$

(2) A *-homomorphism $\phi\colon \oplus_{i=1}^n M_{n_i}(C(X_i)) \to M_k(Y)$ is said to be *diagonal* if there exists natural numbers $k_1, \ldots, k_n$ such that $\sum_i k_i = k$ and $n_i | k_i$, an embedding $\iota\colon \oplus_{i=1}^n M_{k_i} \mapsto M_k$, and diagonal maps $\phi_i\colon M_{n_i}(C(X_i)) \to M_{k_i} \otimes C(Y)$ such that $\phi = \oplus_{i=1}^n \phi_i$.

Definition

(3) A unital *-homomorphism

$$\phi\colon \oplus_{i=1}^{n} M_{n_i}(C(X_i)) \to \oplus_{j=1}^{m} M_{m_j}(C(Y_j))$$

is *diagonal* if each restriction

$$\phi_j\colon \oplus_{i=1}^{n} M_{n_i}(C(X_i)) \to M_j(C(Y_j))$$

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Definition

An inductive limit algebra $A = \lim_{i\to\infty}(A_i, \phi_i)$ is called *a diagonal AH algebra* if each ϕ_i is unital and diagonal.

LP property for diagonal AH algebras

Note that the class of diagonal AH algebras include AF, AI, and AT algebras and Goodearl algebras, and Villadsen algebras of the first type. It also contains important algebras in the Classification Theorem which have the same K-groups and tracial data but diffirent Cuntz semigroups.

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Recall that a C*-algebra A has the LP property if the linear span of projections is dense in A.

Theorem (Hoa-Ho-O 2012)

Let A be a diagonal AH algebra. If A has small eigenvalue variation, A has the LP property.

Examples

Theorem (Thomsen '91)

Let $A = \lim_{\rightarrow}(M_{n_i}(C[0,1])), \phi_i)$ be an AI algebra with unital connecting *-homomorphisms ϕ_i.
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- A is a UHF-algebra.
- A has real rank zero.
- A has LP property.
- A has only one trace state.

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Example (Hoa-Ho-O 2012)

There is a simple unital AH algebra B with real rank one (so does not have the LP property) such that $B \otimes \mathcal{K}$ is a diagonal AH-algebra of real rank one with the LP property. Note that $B \otimes \mathcal{K}$ does not have the small eigenvalue variation.

Example (Elliott '93)

There is an outer action α of $\mathbf{Z}/2\mathbf{Z}$ on CAR such that $CAR \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ is an AI algebra of real rank one.

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Question

1. When is the LP property stable under the crossed products ?
2. More general, let $P \subset A$ be an inclusion of unital C*-algebras. If A has the LP property, when does P have the LP property ?

Basic facts for LP property

Lemma (Pedersen '80)

Let A be a simple C*-algebra with a non-trivial projection. If A has no tracial state or has a unique tracial state then A has the LP property.

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Example

Let $\mathcal{Z}$ be the Jiang-Su algebra. Then $\mathcal{Z}$ does not have the LP-property. From the above lemma $M_2(\mathcal{Z})$ has the LP property. This implies that the LP property is not stable under the hereditary.

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Elliott's example $A = CAR \rtimes_{\alpha} \mathbf{Z}/2\mathbf{Z}$ shows that the LP property is not stable under crossed products in general.
This also gives that the LP property is not stable under the fixed point algebras. Indeed, $A \rtimes_{\hat{\alpha}} \mathbf{Z}/2\mathbf{Z}$ is CAR algebra, so has the LP property. But the fixed point algebra $(A \rtimes_{\hat{\alpha}} \mathbf{Z}/2\mathbf{Z})^{\hat{\hat{\alpha}}}$ is A, that is, it does not have the LP property.

Main Results

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For a C*-algebra A we set

$$C_0(A) = \{(a_n) \in \ell^\infty(\mathbf{N}, A) \colon \lim_{n\to\infty} \|a_n\| = 0|\},$$
$$A^\infty = \ell^\infty(\mathbf{N}, A)/C_0(A).$$

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Definition (Izumi 2004)

Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rokhlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A' \cap A^\infty$ consisting of projections satisfying $(\alpha_g)_\infty(e_h) = e_{gh}$ for $g, h \in G$. We call $\{e_g\}_{g\in G}$ Rokhlin projections.

Theorem (Hoa-Ho-O 2012)

Let A be a simple unital C*-algebra and α be an action of a finite group G. Suppose that α has the Rokhlin property. We have, then, that if A has the LP property, the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ have the LP property.

Theorem (Hoa-Ho-O 2012)

Let A be a simple unital C*-algebra and α be an action of a finite group G. Suppose that α has the Rokhlin property. We have, then, that if A has the LP property, the fixed point algebra A^{α} and the crossed product $A \rtimes_{\alpha} G$ have the LP property.

To get the main result we do not use the local property for the crossed product algebra $A \rtimes_{\alpha} G$ in the sense of C. N. Phillips; for every finite subset $S \subset A \rtimes_{\alpha} G$ and every $\varepsilon > 0$ there are n, a projection $f \in A$, and a unital *-homomorphism $\phi\colon M_n \otimes fAf \to A \rtimes_{\alpha} G$ such that $\mathrm{dist}(a, \phi(M_n \otimes fAf)) < \varepsilon$ for all $a \in S$.

Inclusions of C*-algebras

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Definition

Let $A \supset P$ be an inclusion of unital C^*-algebras with a conditional expectation E from A onto P.

1. A *quasi-basis* for E is a finite set $\{(u_i, v_i)\}_{i=1}^n \subset A \times A$ such that for every $a \in A$,

$$a = \sum_{i=1}^n u_i E(v_i a) = \sum_{i=1}^n E(a u_i) v_i.$$

2. When $\{(u_i, v_i)\}_{i=1}^n$ is a quasi-basis for E, we define $\mathrm{Index} E$ by

$$\mathrm{Index} E = \sum_{i=1}^n u_i v_i.$$

   When there is no quasi-basis, we write $\mathrm{Index} E = \infty$. $\mathrm{Index} E$ is called the Watatani index of E.

Definition (Kodaka-Osaka-Teruya:2008)

A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index} E)^{-1} \cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

Proposition (Kodaka-Osaka-Teruya 2008)

Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{\#G} \sum_{g \in G} \alpha_g(x) \quad \text{for } x \in A,$$

where $\#G$ is the order of G. Then α has the Rohlin property if and only if there is a projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{\#G} \cdot 1$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

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Remark

In the above Proposition if A is simple, an action α has the Rokhlin property if and only if the canonical conditional expectation E has the Rokhlin property.

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Proposition (Kodaka-Osaka-Teruya 2008)

Let $P \subset A$ be an inclusion of unital C^*-algebras and E be a conditional expectation from A onto P with a finite index. If E has the Rokhlin property with a Rokhlin projection $e \in A_\infty$, then there is a unital linear map $\beta\colon A^\infty \to P^\infty$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty$. In particular, $\beta_{|_A}$ is a unital injective *-homomorphism and $\beta(x) = x$ for all $x \in P$.

Theorem

Let $1 \in P \subset A$ be an inclusion of unital C^*-algebras with a finite Watatani index and $E\colon A \to P$ be a faithful conditional expectation. Suppose that A has the LP property and E has the Rokhlin property. Then P has the LP property.

LP property for crossed products

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Let α be an action of a finite group G on a simple unital C*-algebra A which has the LP property. Suppose that α has the Rokhlin property.

LP property for crossed products

Let α be an action of a finite group G on a simple unital C*-algebra A which has the LP property. Suppose that α has the Rokhlin property.
Consider the following two basic constructions:

$$A^\alpha \subset A \subset C^*\langle A, e_P\rangle \subset C^*\langle B, e_A\rangle \ (B = C^*\langle A, e_P\rangle)$$
$$(A^\alpha) \subset A \subset A \rtimes_\alpha G \subset C^*\langle A \rtimes_\alpha G, e_F\rangle,$$

where $F\colon A \rtimes_\alpha G \to A$ is a canonical conditional expectation.
Then there is an isomorphism $\pi\colon C^*\langle A, e_P\rangle \to A \rtimes_\alpha G$ and $\tilde{\pi}\colon C^*\langle B, e_A\rangle \to C^*\langle A \rtimes_\alpha G, e_F\rangle$.

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$$(A^{\alpha}) \subset A \subset A \rtimes_{\alpha} G \subset C^*\langle A \rtimes_{\alpha} G, e_F\rangle,$$

Since canonical conditional expectation $E\colon A \to A^{\alpha}$ has the Rokhlin property, A^{α} has the LP property.
Note that $C^*\langle A \rtimes_{\alpha} G, e_F\rangle \cong M_{|G|}(A)$. Since $M_{|G|}(A)$ has the LP property, so does $C^*\langle A \rtimes_{\alpha} G, e_F\rangle$, hence $C^*\langle B, e_A\rangle$ has the LP property. Since the double conditional expectation $\hat{\hat{E}}\colon C^*\langle B, e_A\rangle \to C^*\langle A, e_P\rangle$ has the Rokhlin property, we know that $C^*\langle A, e_P\rangle$ has the LP property. Therefore, $A \rtimes_{\alpha} G$ ($\cong C^*\langle A, e_P\rangle$) has the LP property.

More example

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Definition (N. C. Phillips 2006)

Let A be an infinite dimensional simple C^*-algebra and let α be an action from a finite group G on $\mathrm{Aut}(A)$. Recall that α has *the tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$ and all $a \in F$,

(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

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Proposition

Let α be an action of a finite group G on a simple unital C^*-algebra A with a unique tracial state. Suppose that α has the tracial Rokhlin property. If A has the LP property, then the crossed product $A \rtimes_\alpha G$ has the LP property.

Thanks !

謝謝！